Theory of Computing

Lecture 2

MAS 714

Hartmut Klauck

Sorting

- Computers spend a lot of time sorting!
- Assume we have a list of numbers $x_1,\ldots,x_n$ from a universe U
- For simplicity assume the x_i are distinct
- The goal is to compute a permutation π such that $x_{\pi(1)} < x_{\pi(2)} < \cdots < x_{\pi(n)}$
- Think of a deck of cards
- Simple model:
  - Input: permutation of 1,...,n
  - Output: inverse permutation

Last week: Binary Insertion Sort

- Using binary search in InsertionSort reduces the number of comparisons to O(n log n)
  - The outer loop is executed n times, each inner loop now uses log n comparisons
- Unfortunately the number of swaps does not decrease:
  - To insert an element we need to shift the remaining array to the right!

Quicksort

- Quicksort follows the „Divide and Conquer“ paradigm
- The algorithm is best described recursively

- Idea:
  - Split the sequence into two
    - All elements in one sequence are smaller than in the other
  - Sort each sequence recursively
  - Put them back together

Quicksort

- Quicksort(A,l,r)
  - If l=r return A
  - Choose a pivot position j between l and r
  - u=1,v=1, initialize arrays B,C
  - for (i=l…r): If A[i]<A[j] then B[u]=A[i], u++
If A[i]>A[j] then C[v]=A[i], v++
  - Run Quicksort(B,1,u) and Quicksort(C,1,v) and return their output (concatenated), with A[j] in the middle

How fast is it?

- The quality of the algorithm depends on how we split up the sequence
- Intuition:
  - Even split will be best
- Questions:
  - How are the asymptotics?
  - Are approximately even splits good enough?

Worst Case Time

- We look at the case when we really just split into the pivot and the rest (maximally uneven)
- Let $T(n)$ denote the number of comparisons for n elements
- $T(2)=1$
- $T(n) <= T(n-1)+n-1$
- Solving the recurrence gives $T(n)=O(n^2)$

Best Case Time

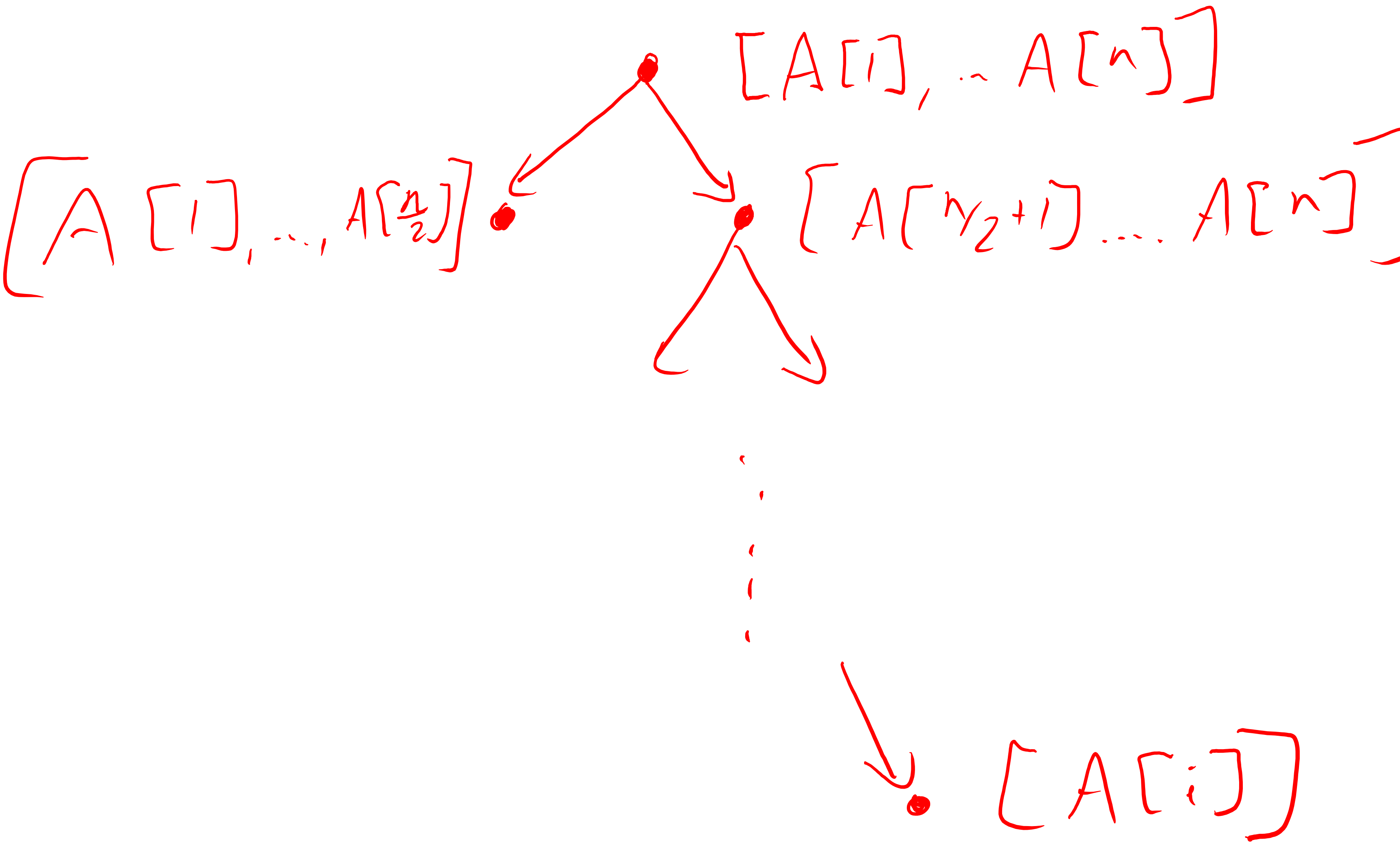
- Best: every pivot splits the sequence in half

- $T(2)=1$
- $T(n)=2T(n/2)+n-1$

- Questions:
  - How to solve this?
  - What if the split is 3/4 vs. 1/4 ?

Recurrences

- How to solve simple recurrences
- Several techniques
- Idea: Consider the recursion tree
- For Quicksort every call of the procedure generates two calls to a smaller Quicksort procedure
  - Problem size 1 is solved immediately
- Nodes of the tree are labelled with the sequences that are sorted at that node
- The cost of a node is the number of comparisons used to split the sequence at the node. I.e. is equal to the length of the sequence at the node minus 1.



[A[1], .. A[n]]
[A[1], ..., A[n/2]]
[A[n/2+1] .... A[n]]
[A[i]]

Example: the perfect tree

- In the best case the sequence length halves-> after log n calls the sequence has length 1
- Depth of the tree is log n
  - Number of nodes is O(n)
  - But each node has a cost
    - nodes on level 0 cost n-1, on level 1 cost n/2-1 etc.
      - For simplicity we assume n is a power of 2
  - Level i has 2^i nodes of cost $n/2^i - 1$
- Total cost is O(n) per level-> O(n log n)

Verifying the guess

- Guess: $T(n) <= n \log n$
- $T(1)$: $T(1)=0 = 1 \log (1)$
- $T(2)$: $T(2) = 1 <= 2$
- $T(n) <= 2T(n/2) + n-1$

$$
\begin{aligned}
&<= n \log(n/2) + n-1 \\
&<= n \log n - n + n-1 \\
&<= n \log n
\end{aligned}
$$

The Master Theorem

- The Master Theorem is a way to get solutions to recurrences
- **Theorem:**
  a,b constant, f(n) function
  Recurrence T(n)=a T(n/b) + f(n)
  and T(O(1)) constant

- 1) $f(n) = O(n^{\log_b a - \varepsilon}) \Rightarrow T(n) = \Theta(n^{\log_b a})$
- 2) $f(n) = \Theta(n^{\log_b a}) \Rightarrow T(n) = \Theta(n^{\log_b a} \cdot \log n)$
- 3) $f(n) = \Omega(n^{\log_b a + \varepsilon}) \Rightarrow T(n) = \Theta(f(n))$

The Master Theorem

- We omit the proof
- Application:
- $T(n)=9T(n/3)+n$
- $a=9$, $b=3$, $f(n)=n$, $n^{\log_b(a)}=n^2$
  - Case 1 applies, Solution is $T(n)=O(n^2)$

Attempt on the case of uneven splits

- Assume every pivot splits exactly ¾n vs. n/4
  - $T(n)=T(3n/4)+T(n/4)+n$
- Same idea:
  - Nodes on level i have cost at most $n/(3/4)^i$
  - There are at most $\log_{4/3} n$ levels
  - What is the total cost of all nodes at a level?
  - Note that all K nodes on a level correspond to a partition of all n inputs into K sets!
  - If sets have size s1,…, sK then comparisons are s1-1+ … sK-1
  - less than n comparisons on any one level

Quicksort Time

- So if every split is partitioning the sequence somewhat evenly (99% against 1%) then the running time is O(n log n)

Average Case Time

- Suppose the pivot is chosen in any fixed way
  - Say, the first element
- Claim: the expected running time of Quicksort is O(n log n)
  - Expected over what?
  - Chosing a *random* permutation as the input
    - Recall that the input to the sorting problem is a permutation

Average Time

- Intuition: Most of the time the first element will be in the “middle” of the sequence for a random permutation
  - Most of the time we have a (quite) balanced split
- Constant probability of an uneven split
  - Can increase running time by a constant factor only
    - Assume nothing gets done on those splits
    - “Merge” balanced and unbalanced splits

Average Time

- **Theorem:**
  On a uniformly random permutation/input the expected running time of Quicksort is
  O(n log n)

Note of Caution

- For any fixed (simple) pivoting rule there are still permutations that need time n^2
  - e.g. pivot is always the minimum
- How to fix this?
- Choose pivot such that the algorithm behaves in the same way as for a random permutation!

Randomized Algorithms

- A randomized algorithm is an algorithm that has access to a source of random numbers
- Different types:
  - Measure expected running time
    - with respect to the random numbers, NOT the inputs
  - Allow errors with some small probability
- We will (for now) consider the first type

Randomized Quicksort

- Use the standard Quicksort,
- BUT choose a random position between l and r as the pivot

- **Theorem:** Randomized Quicksort has (expected) running time O(n log n)

Average Time

- The theorem about randomized Quicksort implies the theorem about the average case time bound for deterministic Quicksort
- Reason:
  - In any partition step the first element of a random permutation and and a random element for a fixed permutation behave in the same way

Proof

- Proof (randomized Quicksort)
- We will count the expected number of comparisons
- Denote by X_{ij} the indicator random variable that is 1 if x_i is compared to x_j
  - At any time
  - x_i is the ith element of the sorted sequence
- Note that all comparisons involve the pivot element

Proof

- The expected number of comparisons is
  $E[\sum_{i=1\ldots n-1} \sum_{j=i+1\ldots n} X_{ij}]$
  $= \sum_{i=1\ldots n-1} \sum_{j=i+1\ldots n} E[X_{ij}]$
- $E[X_{ij}]$ is the probability that x_i is compared to x_j
- Z_{ij} is the set of keys between x_i and x_j
- Claim:
  x_i is compared with x_j iff x_i or x_j is the first pivot chosen among the elements of Z_{ij}

Proof

- Pivots are random, i.e.,
  Prob(x_i first pivot in Z_{ij})=1/(j-i+1)
- $E[X_{ij}] = 2/(j-i+1)$
- Number of comparisons:
  $\sum_{i=1\ldots n-1}\sum_{j=i+1\ldots n} E[X_{ij}]$
  $= 2\sum_{i=1\ldots n-1}\sum_{j=i+1\ldots n} 1/(j-i+1)$
  $= 2\sum_{i=1\ldots n-1}\sum_{k=1\ldots n-i} 1/(k+1)$
  $< 2\sum_{i=1\ldots n-1}\sum_{k=1\ldots n} 1/k$
  $=O(n \log n)$ [Harmonic Series]

Proof

- Hence the expected number of comparisons is $O(n \log n)$
- Easy to see that also the running time is $O(n \log n)$

How fast can we sort?

- There are deterministic algorithms that sort in worst case time $O(n \log n)$
- Do better algorithms exist?
  - Example [Andersson et al. 95]:
    On a unit cost RAM, word length w, one can sort n integers in the range $0 \ldots 2^w$ in time $O(n \log\log n)$
    Even in $O(n)$ if $w > \log^2 n$
  - Not comparison based!

A lower bound

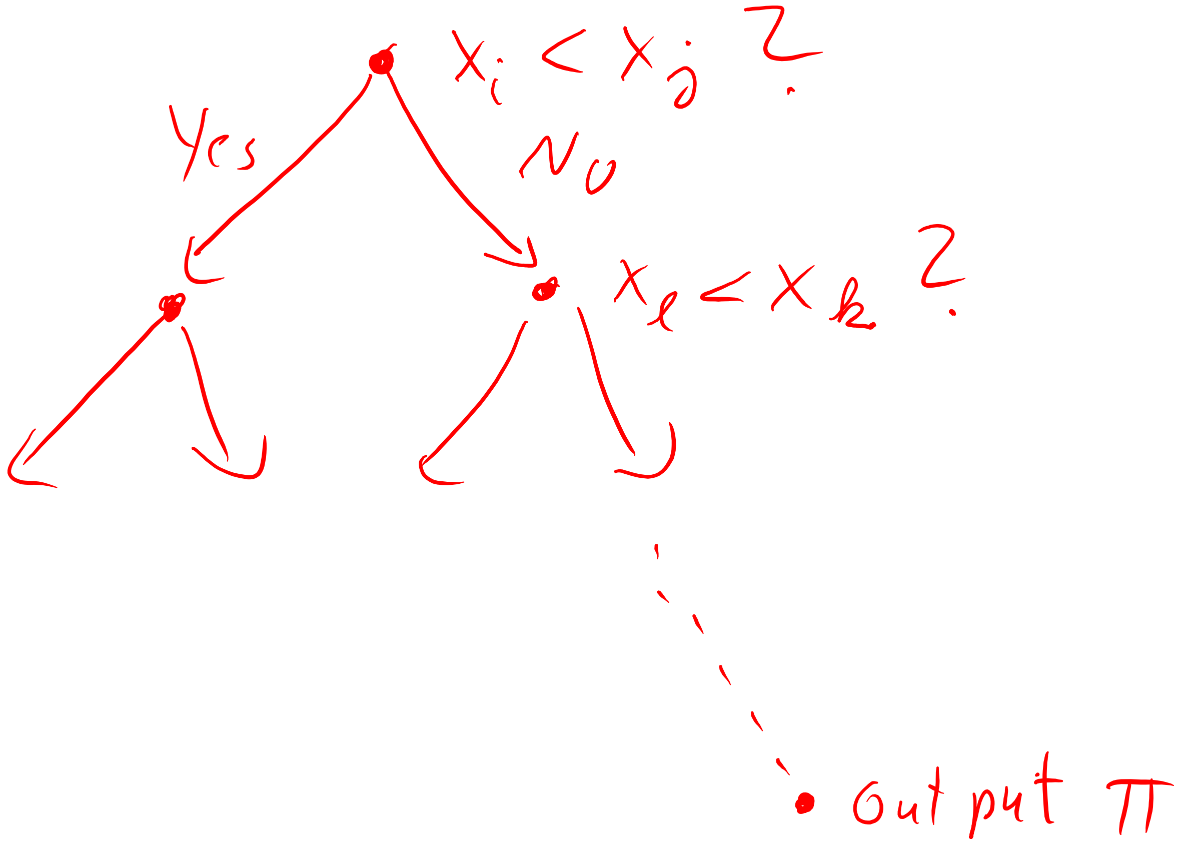
- Assume an algorithm uses only comparisons to access the inputs
  - I.e., it can compare A[i] and A[j] but CANNOT use the sum of A[i] and A[j] or compute A[A[i]]

- **Theorem:** Any comparison based sorting algorithm has to make Ω(n log n) comparisons

Proof:

- We model every algorithm using comparisons by a *Decision Tree*
- Decision Trees can access comparisons and branch according to the result
- Leaves must display the result
  - In our case the result is the permutation needed to sort the input sequence

Example



Proof

- Any algorithm based on comparisons can be simulated by a decision tree
- The input to the decision tree is the table of n^2 comparisons
  - at position (i,j) the table contains 1 if $x_i<x_j$ and 0 if $x_i \geq x_j$
  - A decision tree is an algorithm that can query the comparison table at any position (based on the results of previous queries)
  - The leafs of the decision tree are labelled with permutations π that sort the input sequence
  - The depth of the tree is the max number of comparisons made during any computation

Proof

- **Fact:** There are n! different permutations of n elements
- **Lemma:** Any decision tree for sorting must have n! Leaves
  - Every permutation leads to a different output
- **Lemma:** Any binary tree with n! leaves must have depth at least $\log(n!) = \Omega(n \log n)$

- **Conclusion:** Any DT for sorting has depth $\Omega(n \log n)$.
- And any comparison based algorithm needs $\Omega(n \log n)$ comparisons to sort